

BOUND FOR THE MAXIMAL PROBABILITY IN THE LITTLEWOOD–OFFORD PROBLEM

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ABSTRACT. The paper deals with studying a connection of the Littlewood–Offord problem with estimating the concentration functions of some symmetric infinitely divisible distributions. It is shown that the values at zero of the concentration functions of weighted sums of i.i.d. random variables may be estimated by the values at zero of the concentration functions of symmetric infinitely divisible distributions with the Lévy spectral measures which are multiples of the sum of delta-measures at \pm weights involved in constructing the weighted sums.

The concentration function of a \mathbf{R}^d -dimensional random vector Y with distribution $F = \mathcal{L}(Y)$ is defined by the equality

$$Q(F, \tau) = \sup_{x \in \mathbf{R}^d} \mathbf{P}(Y \in x + \tau B), \quad \tau \geq 0,$$

where $B = \{x \in \mathbf{R}^d : \|x\| \leq 1/2\}$ is the centered ball of the Euclidean space \mathbf{R}^d of radius $1/2$. In particular,

$$Q(F, 0) = \sup_{x \in \mathbf{R}^d} \mathbf{P}(Y = x).$$

Let X, X_1, \dots, X_n be independent identically distributed random variables. Let $a = (a_1, \dots, a_n)$, where $a_k = (a_{k1}, \dots, a_{kd}) \in \mathbf{R}^d$, $k = 1, \dots, n$. Starting with seminal papers of Littlewood and Offord [11] and Erdős [9], the behavior of the concentration functions of the weighted sums $S_a = \sum_{k=1}^n X_k a_k$ is studied intensively. In the sequel, let F_a denote the distribution of the sum S_a . The first version of the Littlewood–Offord problem was related to the estimation of $Q(F_a, 0)$ in the case where X has the symmetric Bernoulli distribution $\mathbf{P}(X = 1) = \mathbf{P}(X = -1) = 1/2$.

In the last ten years, refined concentration results for the weighted sums S_a play an important role in the study of singular values of random matrices (see, for instance, Nguyen and Vu [12, 13], Rudelson and Vershynin [14, 15], Tao and Vu [16, 17], Vershynin [18], Later, somewhat different bounds for the concentration functions in the Littlewood–Offord problem were obtained by Eliseeva, Götze and Zaitsev [4]–[8]. The aforementioned results reflect

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the dependence of the bounds on the arithmetic structure of coefficients a_k under various conditions on the vector $a \in (\mathbf{R}^d)^n$ and on the distribution $\mathcal{L}(X)$.

Below, in some formulas, the quantity

$$p(v) = G\{\{x \in \mathbf{R} : |x| > v\}\}, \quad v \geq 0,$$

will appear, where G is the symmetrized distribution $G = \mathcal{L}(X_1 - X_2)$. Writing $A \ll_d B$ means that $|A| \leq c(d)B$, where $c(d) > 0$ depends on d only. The inner product of vectors $x, y \in \mathbf{R}^d$ is written as $\langle x, y \rangle$. Denote by \mathfrak{F}_d the set of all d -dimensional probability distributions and by \mathfrak{D}_d the set of all infinitely divisible distribution from \mathfrak{F}_d . We denote by E_y the distribution concentrated at a point $y \in \mathbf{R}^d$. Below $\widehat{F}(t)$, $t \in \mathbf{R}^d$, is the characteristic function of $F \in \mathfrak{F}_d$.

Products and powers of measures will be understood in the sense of convolution. For $D \in \mathfrak{D}_d$, $F \in \mathfrak{F}_d$, and $\lambda \geq 0$, by D^λ we denote infinitely divisible distribution with the characteristic function $\widehat{D}^\lambda(t)$, and by $e(\lambda F)$ infinitely divisible compound Poisson distribution with the characteristic function $\exp(\lambda(\widehat{F}(t) - 1))$, and with the Lévy spectral measure λF . It is easy to see that

$$e(\lambda F) = e^{-\lambda} \sum_{s=0}^{\infty} \frac{\lambda^s F^s}{s!}.$$

Here F^0 is the degenerate distribution E_0 concentrated at the origin $0 \in \mathbf{R}^d$. Thus,

$$e(\lambda F) = e^{-\lambda} E_0 + e^{-\lambda} \sum_{s=1}^{\infty} \frac{\lambda^s F^s}{s!}. \quad (1)$$

We need some simplest properties of concentration functions. First we note that for any distribution $F \in \mathfrak{F}_d$

$$\lim_{\tau \rightarrow 0} Q(F, \tau) = Q(F, 0) \quad (2)$$

(see, e.g., [10, p. 14]). For any $U, V \in \mathfrak{F}_d$, we have

$$Q(UV, \tau) \leq Q(U, \tau), \quad \text{for all } \tau \geq 0. \quad (3)$$

The following Lemma 1 follows directly from (1) and (3).

Lemma 1. *Let $\tau, \lambda \geq 0$, $F, U \in \mathfrak{F}_d$ and $D = e(\lambda F) \in \mathfrak{D}_d$. Then*

$$0 \leq Q(U, \tau) - Q(UD, \tau) \leq 1 - e^{-\lambda}. \quad (4)$$

Introduce the distribution H , with the characteristic function

$$\widehat{H}(t) = \exp\left(-\frac{1}{2} \sum_{k=1}^n (1 - \cos\langle t, a_k \rangle)\right), \quad t \in \mathbf{R}^d.$$

Note that $H^p = e\left(\frac{np}{2}M\right)$ is a symmetric infinitely divisible distribution with the Lévy spectral measure $\frac{np}{2}M$, where

$$M = \frac{1}{2n} \sum_{k=1}^n (E_{a_k} + E_{-a_k}) \in \mathfrak{F}_d.$$

M. A. Lifshits drew the author's attention to the fact that the distribution H^p can be represented as the distribution of weighted sum S_a for the same set of weights $a = (a_1, \dots, a_n)$ which is involved in the the original problem. But the common distribution of the random variables X, X_1, \dots, X_n , for fixed p , has a special form $\mathcal{L}(X) = e\left(\frac{p}{4}(E_1 + E_{-1})\right)$.

The following Theorem 1 is contained in the recently published paper [8], see also [6]. It connects the Littlewood–Offord problem with general bounds for the concentration functions, in particular, with the results of Arak contained in [1]–[3] (see [6]).

Theorem 1. *For any $\tau, u > 0$, the inequality*

$$Q(F_a, \tau) \ll_d Q(H^{p(\tau/u)}, u)$$

holds.

In a recent paper of Eliseeva and Zaitsev [8], a more general statement than Theorem 1 is obtained. It gives useful bounds if $p(\tau/u)$ is small, even if $p(\tau/u) = 0$.

Theorem 1 has been proved for $\tau, u > 0$. The naturally arising question is to find an analogue of Theorem 1 for $\tau = 0$. The answer to this question is given by the following Theorem 2.

Theorem 2. *The inequality*

$$Q(F_a, 0) \ll_d Q(H^{p(0)}, 0) \tag{5}$$

holds.

Let $A = \{x \in \mathbf{R} : \mathbf{P}(X = x) > 0\}$. It is clear that the set A is no more than countable. It is easy to verify that

$$p(0) = 1 - \sum_{x \in A} (\mathbf{P}(X = x))^2.$$

According to (3), $Q(H^{p(0)}, 0)$ is a non-increasing function of $p(0)$. Therefore, the right-hand side of inequality (5) is minimal for $p(0) = 1$. But in this case the left-hand side of (5) vanishes. Indeed, then the distribution $\mathcal{L}(X)$ has no atoms, $A = \emptyset$, and $Q(\mathcal{L}(X), 0) = 0$. Through (3), this easily implies the equality $Q(F_a, 0) = 0$.

The advantage of Theorem 2 is that $p(v)$ takes its maximal value at $v = 0$. It distinguishes Theorem 2 from Theorem 1, which gives for $\tau = u > 0$ the estimate $Q(F_a, \tau) \ll_d Q(H^{p(1)}, \tau)$, which, of course, implies that $Q(F_a, 0) \ll_d Q(H^{p(1)}, 0)$ (see (2)). But $p(0)$ can be substantially greater than $p(1)$ and $Q(H^{p(0)}, 0)$ may be substantially less than $Q(H^{p(1)}, 0)$. Using Theorem 2 instead of Theorem 1 gives a possibility to replace $p(1)$ by $p(0)$ in Theorems 5 and 6 of [6] in a particular case, where the parameters τ_j , $j = 1, \dots, d$, involved in the formulations of these theorems, are all zero.

It is interesting that, in spite of the above, we deduce Theorem 2 still from Theorem 1 with the help of somewhat more delicate passing to the limit.

Proof of Theorem 2. By Theorem 1,

$$Q(F_a, \tau) \ll_d Q(H^{p(\varepsilon)}, \tau/\varepsilon), \quad \text{for all } \tau, \varepsilon > 0.$$

Letting τ to zero, and using (2), we get

$$Q(F_a, 0) \ll_d Q(H^{p(\varepsilon)}, 0), \quad \text{for all } \varepsilon > 0. \quad (6)$$

It is evident that $p(\varepsilon) \leq p(0)$ and $H^{p(0)} = H^{p(\varepsilon)} H^{p(0)-p(\varepsilon)}$. Using inequality (4), we verify the validity of the relation

$$0 \leq Q(H^{p(\varepsilon)}, 0) - Q(H^{p(0)}, 0) \leq 1 - \exp\left(-\frac{1}{2} n(p(0) - p(\varepsilon))\right).$$

Since $p(\varepsilon) \rightarrow p(0)$, this implies that

$$Q(H^{p(\varepsilon)}, 0) \rightarrow Q(H^{p(0)}, 0) \quad \text{as } \varepsilon \rightarrow 0.$$

Letting ε to zero in the right-hand side of inequality (6), we obtain the assertion of Theorem 2.

□

Remark. The weak convergence of probability distributions does not imply, in general, the convergence of values of the concentration functions at zero. For example, this happens when continuous distributions converge to discrete ones.

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